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On the Cognitive Structure of Basic Arithmetic Skills: Operation, Order, and Symbol Transfer Effects

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In 2 experiments, college Ss practiced extensively on single-digit multiplication and division problems (e.g., ___ = 6 × 9; 42 = ___ × 6) and were tested on both practice problems and several altered versions of those problems, which were constructed by changing the required operation, operand order, or arithmetic symbol. There was strong positive transfer to test problems that had exactly the same elements (the numbers and the required operation) as a practice problem, regardless of whether other factors such as operand order or symbol were changed, but little if any positive transfer to test problems that did not have the same elements as a practice problem. An identical elements framework is used to interpret these results and implications for existing computational models of arithmetic fact retrieval and for the development of arithmetic skill are discussed.

Fundamental to any mental calculation is the skill of simple arithmetic; that is, the ability to determine quickly and accurately the answers to problems such as 4 × 7 = ___. In recent years, there has been increasing interest in the cognitive processes underlying this skill in both children and adults (e.g., Ashcraft, 1987; Campbell, 1987a, 1987b, 1987c, 1991; Campbell & Graham, 1985; Fendrich, Healy, & Bourne, 1993; Koshmider & Ashcraft, 1991; McCloskey, Caramazza, & Basili, 1985; McCloskey, Harley, & Sokol, 1991; Miller & Paraedes, 1990; Miller, Perlmutter, & Keating, 1984; Siegler, 1988; Zbordoff & Logan, 1990). A basic finding of this research is that, whereas children often use explicit, consciously mediated counting algorithms, especially in early skill acquisition (e.g., children often solve 3 × 7 by adding 7 + 7 + 7), there is a transition toward retrieval of arithmetic facts directly from memory as skill improves. By adulthood, performance on most single-digit operand problems appears to reflect direct retrieval of facts from memory.

The detailed memory structure of these facts is currently not well understood. It is unclear whether representations are fundamentally bound to the perceptual characteristics of problems, as proposed by Campbell and Clark (1992), or whether they take a more abstract form, as proposed by McCloskey et al. (1985). Embedded within these two general representational schemes are issues concerning the basic "units" into which factual arithmetic knowledge is organized. For example, do complementary operand orders within a commutative operation (e.g., 3 × 8 and 8 × 3) access the same or different underlying memory structures? Similarly, do complementary problems from two operations (e.g., 6 × 7 and 42 ÷ 6) or related problems within a noncommutative operation (e.g., 42 ÷ 6 and 42 ÷ 7) access the same or different memory structures? This issue of what, psychologically speaking, constitutes a unique arithmetic fact is the primary focus of the current study.

These experiments should also have implications for three broader issues. First, several researchers are developing computational models of arithmetic fact retrieval (e.g., Anderson, Spoehr, & Bennett, 1994; Campbell & Oliphant, 1992; McCloskey & Lindemann, 1992; Rickard, Mozer, & Bourne, 1992). These models incorporate contrasting assumptions about the basic units into which arithmetic knowledge is organized, and thus the present experiments should be of value in testing them. Second, the present experiments, and indeed the entire mental arithmetic literature, are motivated by the premise that discoveries about mathematical cognition will have implications for general theories of skill acquisition and memory. Knowledge of calculational thought processes should provide a valuable complement to knowledge of more qualitative, verbal processes, facilitating efforts to construct precise and general theories of basic cognitive processes. Third, a thorough understanding of the structure of adult arithmetic skills should provide a useful guide to future research on both children's performance, and the changes in cognitive structures that occur in transition to adult performance.

Before discussing the present experiments in detail, we review briefly evidence supporting the idea that skilled arithmetic performance is based on fact retrieval (for additional reviews see Ashcraft, 1992; Campbell, 1987a; McCloskey et al., 1991; Rickard et al., 1992). One phenomenon that is consistent with fact retrieval processes is interference among similar items (see Anderson, 1983). This effect has been demonstrated repeatedly in research on adult mental arithmetic. For example, under speeded conditions, typically 70% to 90% of errors that college students make are *table related;* that is, they are answers to problems that share an operand with the problem being solved (e.g., Campbell & Graham, 1985; Gra-

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ham, 1987; Sokol, McCloskey, Cohen, & Aliminosa, 1991). Thus, for instance, 21 (the answer to 3 × 7) is a relatively frequent error to 4 × 7. Similar results have been reported in priming and verification studies (e.g., Campbell, 1987b, 1991; Koshmider & Ashcraft, 1991; Winkelman & Schmidt, 1974). For example, Campbell (1991) presented a double-digit prime (e.g., 24) for 200 ms, followed by a multiplication problem to be solved (e.g., 8 × 4). He found that response times (RTs) were slowest and error rates were highest when the incorrect prime was table related to the problem, with better performance when the prime was *table unrelated* and best performance when the prime was correct.

The convergent theme in most recent theoretical accounts of these and related interference effects (see Anderson et al., 1994; Campbell & Oliphant, 1992; McCloskey & Lindemann, 1992; Rickard et al., 1992) is that (a) skilled performance reflects retrieval of facts from memory, (b) when a retrieval is attempted, multiple facts become active to the extent that they are in some way similar (e.g., share an operand) with the problem or have been primed, and (c) these active representations then compete in an interactive-activation-like process (McClelland & Rumelhart, 1981) until one representation (usually the correct one) reaches a high enough level of activation to be selected as the answer.

Additional support for this general account comes from an investigation by Campbell (1987a) in which college subjects were pretested on a set of simple multiplication problems, then trained for several sessions on a subset of these problems, and then tested again on all problems. With practice, RTs improved considerably. On the posttraining test, subjects performed worse (had significantly slower RTs and a higher error rate) on unpracticed problems than they had on these same problems at pretest. Campbell's finding that practice does not transfer positively across problems within multiplication constitutes important converging evidence for the theoretical claim that skilled arithmetic performance reflects retrieval of individual facts from memory rather than the execution of more general procedures. Furthermore, the finding that practice appears to transfer negatively to unpracticed problems is consistent with the assumption that representations for multiple facts compete for activation during the retrieval process (i.e., the problems on which the subjects practiced compete more strongly after practice, thus slowing RTs and increasing error rates for unpracticed problems).

In another practice–transfer study, Fendrich et al. (1993) trained college students for multiple sessions on simple multiplication problems (e.g., 6 × 8) and then tested them on these same problems, on operand order reverses of these problems (e.g., 8 × 6), and on new problems that were not seen during practice. Like Campbell (1987a), they found that RTs decreased considerably across practice sessions. Also, they found that learning transferred positively, although not completely, to operand order reversed problems, establishing a useful qualifier to the Campbell (1987a) transfer results.

Overview of the Experiments

The studies by both Campbell (1987a) and Fendrich et al. (1993) demonstrate the value of a practice–transfer approach to exploring the structure of basic arithmetic knowledge. Our study is an extension of this general approach. In the first phase of both experiments, subjects were given three sessions of practice on simple multiplication and division problems. Across Experiments 1 and 2, four types of problems were presented at practice; multiplication and division problems with symbol × (e.g., ____ = 4 × 7 and 35 = __ × 5, Experiments 1 and 2), and multiplication and division problems with symbol ÷ (e.g., __ ÷ 6 = 9 and 48 ÷ __ = 6, Experiment 2).[1] We a priori define a problem as involving either multiplication or division according to the arithmetic operation that is formally required to produce the answer. Whether this classification is also appropriate as a description of the cognitive organization of arithmetic knowledge is one of the empirical questions that we address.

The primary purpose of practice was to permit evaluation of transfer of learning to the various altered problems in the test phase of the experiment. The practice data also allowed us to explore two secondary issues. First, they provided a test of the applicability of the power law of practice (Newell & Rosenbloom, 1981), which has been demonstrated previously for mental addition (Crossman, 1959) and multiplication (Charness & Campbell, 1988; Staszewski, 1988), but not yet for division. Second, they allowed us to determine the relative difficulty of multiplication and division problems and of problems expressed with the symbols × and ÷.

In the second, test phase of the experiments, subjects solved each of the problems seen at practice, as well as several altered versions of each of the practice problems. Table 1 lists the test conditions in Experiment 1 and provides an example of each. Multiplication and division problems are presented separately because the results depended critically on this distinction.

The results of the test phase should provide important new information concerning the basic knowledge units into which arithmetic facts are organized. Here, the term *knowledge unit* is used generically to refer to the combination of representation and process that governs performance on a given problem, or group of problems. It is generally believed that a given set of empirical results can be modeled by multiple theories that make widely varying assumptions about process and representation (e.g., Anderson, 1992). As such, the current experiments are unlikely to yield unique specifications of process or representation in isolation. Nevertheless, given that relatively good transfer of learning from practice to test provides evidence of access to a common knowledge unit and that relatively poor transfer provides evidence of access to different knowledge units, the test results from Experiment 1 will provide evidence bearing on the following basic questions: Do common or different knowledge units underlie performance on (a) complementary operand orders within a commutative operation (e.g., ____ = 4 × 7 and ____ = 7 × 4), (b) related problems within a noncommutative operation (e.g., 28 = __ × 4

[1] All problems were formatted with the product to the left (instead of the more traditional format, in which the product is to the right) to match the stimulus items of Experiments 1 and 2 (which were designed at the same time), as closely as possible. The product could only be to the left in Experiment 2 because the symbol "÷" was used in the context of multiplication.

and 28 = __ × 7), and (c) complementary multiplication and division problems (e.g., ____ = 7 × 4 and 28 = __ × 4)?

These data will in turn be useful in evaluating computational models of mental arithmetic, in which assumptions about both process and representation must be made explicit. Consider, for example, the network-interference model developed recently by Campbell and Oliphant (1992). This model assumes that problems are represented in specific "physical codes" that are tied to the perceptual modality through which the problems are processed. A problem presented in a visual format, as in our experiments, maps onto a specific visual code for that problem. For example, the problem ____ = 7 × 8 maps onto the representation {7, 8; ×; 56}. The basic unit of representation is an ensemble of three subunits; one for the pair of operands, one for the operation symbol, and one for the answer. The order and perceptual characteristics of the features within a subunit are preserved in the representation. One implication of this model is that there are separate representations for complementary operand orders. For example, the problem ____ = 6 × 8 maps onto the representation {6, 8; ×; 48}, whereas the problem ____ = 8 × 6 maps onto the representation {8, 6; ×; 48}. In general terms, the retrieval process in the network-interference model involves (a) excitation of problem nodes according to their feature overlap with the presented problem, (b) inhibition among problem nodes resulting in low activation levels for nodes that do not have strong feature overlap with the problem, and (c) excitation from problem nodes to candidate answer nodes in a separate response production system.

Although the network-interference model is currently developed for multiplication and addition, Campbell and Oliphant (1992) suggest that one straightforward way to generalize their model to division would be to assume that a single representation supports performance on complementary multiplication and division problems. Thus, for example, performance on both ____ = 5 × 6 and 30 = __× 6 could potentially be mediated in their model by a single representation, {5, 6; ×; 30}. Because speedup with practice in the model is due primarily to strengthening of a single unitization parameter associated with each representation (Campbell, 1992), this version of the model would predict substantial transfer of learning to operation change problems (multiplication to division and vice versa) in Experiment 1. Also, performance on operation change problems should be better than performance on either operand order change or operation and operand order change problems because problems in these later two conditions cannot access the representation that was strengthened during practice as efficiently as can problems in the operation change condition. Verification of the above predictions in the current experiments would provide strong support for the form of representation adopted by Campbell and Oliphant (1992). Failure to verify these predictions (e.g., if performance in the operand order change condition is better than performance in the operation change condition) would indicate that some elaboration of the Campbell and Oliphant representational scheme will be necessary to extend the model to division.

The network-interference model also appears to predict some positive transfer of learning to multiplication operand order change problems at test. Representations for both orders of a problem will receive some excitatory activation when either problem is presented. Also, representations for both orders are, of course, associated with the same response. Thus, performance on problems in the operand order change condition at test should benefit from practice that would strengthen the representation for the practiced operand order. What is less clear, and cannot be determined without performing a simulation, is the degree of transfer predicted by the model. The data on operand order transfer, then, will provide a set of empirical results against which the model can eventually be tested in quantitative detail.

Table 1
Test Conditions for an Example Problem in Experiment 1

Test condition	Practice	Test
	Multiplication	
No change	____ = 4 × 7	____ = 4 × 7
Operand order change	____ = 7 × 4	____ = 4 × 7
Operation change	28 = __ × 7	____ = 4 × 7
Operation and operand order change	28 = __ × 4	____ = 4 × 7
	Division	
No change	28 = __ × 7	28 = __ × 7
Operand order change	28 = __ × 4	28 = __ × 7
Operation change	____ = 4 × 7	28 = __ × 7
Operation and operand order change	____ = 7 × 4	28 = __ × 7

It is worth noting that in the current experiments, subjects first saw the problems on the computer screen without the answer (e.g., ____ = 4 × 7) and then, after they responded, the answer they produced (which was the correct answer on the vast majority of trials) appeared on the screen with the rest of the problem (e.g., 28 = 4 × 7). This same sequence of events was repeated for each problem many times across the practice sessions. Thus, these experiments would appear to provide an ideal learning context favoring the type of perceptually specific, unitized problem–answer representations assumed in the network-interference model.

Another recent computational model of mental multiplication, MATHNET (McCloskey & Lindemann, 1992), also makes some predictions about test performance in Experiment 1, at least with respect to the operand order change condition for multiplication. MATHNET is a connectionist model, incorporating the mean field theory learning algorithm (Hinton, 1989) that appears to learn an order-specific form of representation in a hidden layer. McCloskey and Lindemann reported that when the network is trained and then damaged (by randomly decrementing some of the weight values), there is little, if any, evidence of a relationship between complementary operand orders in degree of impairment. In one simulation, to take an extreme example, damage resulted in a 100% error rate for 7 × 6, and a 0% error rate for 6 × 7. McCloskey and Lindemann suggested that there may be a slight positive correlation in error rate for complementary operand orders, but they presented no quantitative evidence to support this claim. To a good first-order approximation then, MATHNET appears to learn separate representations for complementary

Table 2
A Sample Practice Set (Practice Set 1) Used in Experiment 1

Multiplication	Division
____ = 6 × 9	36 = __ × 4
____ = 7 × 9	27 = __ × 9
____ = 4 × 8	48 = __ × 8
____ = 8 × 7	42 = __ × 7
____ = 8 × 9	28 = __ × 4
____ = 7 × 5	40 = __ × 8
____ = 6 × 5	45 = __ × 9
____ = 2 × 8	21 = __ × 7
____ = 5 × 4	14 = __ × 7
____ = 3 × 5	10 = __ × 5
____ = 3 × 4	12 = __ × 2
____ = 6 × 3	18 = __ × 9
____ = 8 × 3	24 = __ × 6
____ = 4 × 2	6 = __ × 2
____ = 1 × 6	8 = __ × 8
____ = 9 × 1	7 = __ × 1
____ = 1 × 4	3 = __ × 1
____ = 5 × 1	2 = __ × 1

operand orders in multiplication. Given this aspect of the model, it is reasonable to assume that if the model were first taught both operand orders of a set of multiplication problems (roughly to simulate real-world learning of arithmetic facts) and then trained further on only one order of each problem (to simulate the current experiments), the effects of practice would not transfer to the unpracticed operand order.

We note that the examples above are included primarily as illustrations of the importance of the issues we are investigating to current theoretical efforts in the area of mental arithmetic. We do not mean to advertise the experiments as providing definitive tests of MATHNET, the network-interference model, or any other proposed model, although we do believe that the experiments (and practice–transfer approach in general) can yield results that will prove valuable both in testing aspects of these models as currently formulated and in guiding their future development.

Experiment 1

Method

Apparatus and materials. Subjects were tested on Zenith Data Systems personal computers, programmed with the Micro Experimental Language (MEL) software (Schneider, 1988). Four practice sets were constructed to allow counterbalancing across operand order and operation and to control for possible effects of ascending-descending operand order and problem difficulty. An example practice set (Practice Set 1) is shown in Table 2. The practice sets were constructed in the following manner: Excluding squares problems (e.g., 4 × 4), and considering for the moment complementary operand orders to be separate problems, there are 72 problems between 2 × 1 and 9 × 8, inclusive. These problems were divided into two subsets of 36 problems (Practice Sets 1 and 2), such that problems differing only in operand order were in different practice sets (i.e., 3 × 9 and 9 × 3 were in different practice sets). Within each of these two practice sets, 18 problems were formatted as multiplication problems and 18 as division problems. The multiplication and division problems in each practice set were roughly equated on problem difficulty. Half of the problems of both operations (multiplication and division) had ascending operand order (e.g., 3 × 6), and half had descending operand order (e.g., 7 × 4). Practice Sets 3 and 4 were then constructed by reversing the operation of each of the problems in Practice Sets 1 and 2 (e.g., if ____ = 5 × 4 was a problem in Practice Set 1, that problem became 20 = __ × 4 in Practice Set 3). In summary, four practice sets were created such that there was exactly one problem from each number relation in each set (e.g., Practice Sets 1, 2, 3, and 4 contained ____ = 6 × 9, ____ = 9 × 6, 54 = __ × 9, and 54 = __ × 6, respectively). Each of these four sets was then used equally often across the 12 subjects during training. The immediate and delayed tests consisted of all 144 problems that made up the four practice sets.

Subjects and procedure. Twelve subjects from an introductory psychology course received credit for participating in the experiment. Each subject was tested for four sessions. Each session lasted about 40 min. The first three sessions were held on Monday, Wednesday, and Friday of one week, and the fourth session was held on Friday 4 weeks later. During practice, each subject was exposed to 40 blocks of problems, 15 in both the first and second sessions, and 10 in the third session. Each block contained one instance of each of the 36 problems in a given subject's practice set. The order in which problems were presented was randomly determined for each block of practice and for each subject. Problems were presented one at a time, centered on the computer screen. As each problem appeared, the subject typed the answer using the numeric keypad and then pressed the enter key. As the subject typed the answer, it appeared on the computer screen, replacing the underlined spaces. Subjects were told to answer each problem as quickly and accurately as possible. If the subject entered the correct answer, a "correct answer" notice was displayed below the problem for 1 s. If the subject entered an incorrect answer, an "incorrect answer" notice and the correct answer were displayed below the problem for 1.5 s. In both cases, the screen was then blank for 1 s, and then the next problem was displayed. After each block of 36 trials, a message was displayed on the screen telling subjects that they could rest briefly and requesting subjects to press the enter key to begin the next block.

An immediate test was given at the end of the third session, following the last practice block. Eight blocks of 36 problems were presented in exactly the same manner as the practice problems. Across the first four blocks, each of the 144 problems that made up the four practice sets was presented once. Each of these problems was presented a second time in the final four blocks. Each block contained 9 problems from each of the four practice sets, with each block having an equal number of multiplication and division problems and an equal number of problems with ascending and descending operand orders. Subject to the constraints above, the order of presentation of blocks and the order of problem presentation within each block was determined randomly for each subject. The delayed test was structured in exactly the same way as the immediate test, except that there were twice as many trials so that each of the 144 problems was presented four times across 16 blocks of 36 problems.

Results

The 10 problems with single-digit products (e.g., ____ = 4 × 2; 9 = __ × 1) were analyzed separately from the 26 problems with double-digit products (e.g., ____ = 4 × 7; 56 = __ × 8). This separation was motivated by the possibility that most multiplication problems with single-digit products may be solved by rule (e.g., if a multiplication problem has 1 as one of the operands, the answer is the other operand) rather than by retrieval of facts from memory or any other form of calculation (e.g., McCloskey, Aliminosa, & Sokol, 1991).[2] The results for problems with single-digit products were somewhat difficult to interpret: There was only modest speedup with practice, and differences among the test conditions were small, though reliable in some cases. To the extent that there were reliable effects for these problems, they were analogous to those observed for double-digit problems. For these reasons, the results for problems with single-digit products will not be discussed further. A discussion of these results can be found in the thesis by Rickard (1992).

Unless otherwise indicated, all reported results are reliable at the .05 level. Reported error analyses were performed on the raw error proportions. Secondary analyses on the arcsine transformed error proportions yielded an equivalent set of reliable effects in all instances. Reported RT analyses were performed on the log transformed initiate RT (the interval between the onset of the problem on the computer screen and the pressing of the first digit of the answer) and were limited to correctly solved problems. Previous arithmetic research (Fendrich et al., 1993) showed initiate RT to be highly correlated with total RT, the interval between the onset of the problem and the pressing of the enter key.

Practice. The overall error proportion for multiplication problems was .036 and for division problems, .030. The power law of practice predicts a linear decrease in log RT as a function of log trials or blocks of trials (see Newell & Rosenbloom, 1981). That the practice data essentially conform to this expectation can be seen in Figure 1, which shows log RTs for correctly solved problems plotted by log block and operation (multiplication or division). Each data point represents up to 156 observations; data were collapsed over subjects and problems. A within-subjects regression was performed to confirm the effects of practice (the improvement in log RT as a function of log block) and operation (the advantage of multiplication over division) suggested in Figure 1. The overall r^2 was .90. The effect of log block was strong, $F(1, 11) = 242.6$, $MS_e = .00753$, as was the effect of operation, both at the beginning (first block) of practice, $F(1, 11) = 56.6$, $MS_e = .00449$, and at the end (last block) of practice, $F(1, 11) = 39.1$, $MS_e = .01555$. The antilog RT advantage for multiplication was 341 ms at the beginning of practice and 109 ms at the end of practice. There was also a reliable interaction between operation and log block, $F(1, 11) = 5.8$, $MS_e = .00279$, reflecting a slightly greater rate of speedup for division than for multiplication.

Test. Comparisons of the immediate and delayed test data showed very good retention of improvements in performance levels acquired through practice, a finding that replicates previous results of Fendrich et al. (1993). Each of the reliable effects on the immediate test was also present on the delayed test (although the effect sizes were diminished, likely due to the practice that was received on all test problems during the posttest), and all theoretical conclusions that were supported by the immediate test data were also supported by the delayed test data. Thus, we focused solely on the immediate test results (which we refer to as *test results*). For a discussion of the delayed test results of Experiments 1 and 2, see Rickard (1992). In all test analyses, a block of test problems refers to a group of four actual blocks of problems across which each test problem was presented once (see the *Subjects and procedure* section). Thus, there were two blocks of test problems in this experiment.

Test: Multiplication. As there were no reliable differences in error patterns in Blocks 1 and 2, the error data were

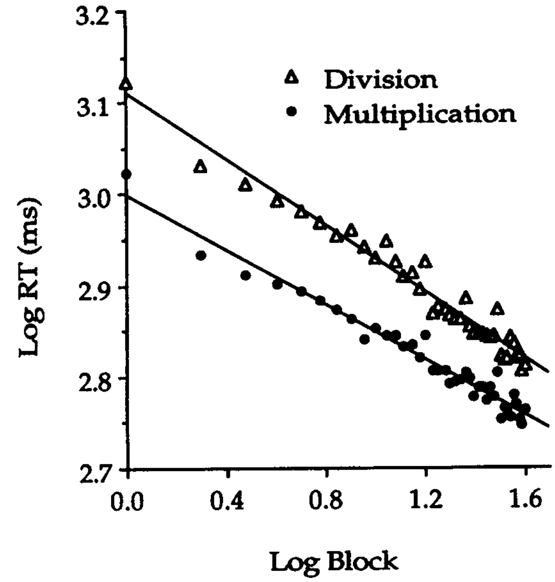

Figure 1. Log response time (RT) for correctly solved practice problems in Experiment 1 plotted as a function of log block and operation (multiplication or division).

collapsed across this variable for analysis. The error proportions for problems in the no-change, operand order change, operation change, and operation plus operand order change conditions were .013, .032, .112, and .115, respectively. A 2 × 2 within-subjects analysis of variance (ANOVA) with operand order (same or different) and operation (same or different) as variables showed a large effect of operation, $F(1, 11) = 12.9$, $MS_e = .01516$, but no reliable effect of operand order and no interaction ($Fs < 1.0$).

The antilog of the mean log initiate RT for multiplication problems (averaged across subjects and correctly solved problems) is plotted by block and test condition (no change, operand order change, operation change, and operation plus operand order change) in Figure 2. Also shown in Figure 2 are the multiplication RTs extrapolated from the power function fits to the practice data (performed separately for each subject), included to allow comparison of these RTs with the actual performance on no-change problems at test.

[2] Some problems with single-digit products did not have 1 as an operand (i.e., 2 × 3 and 2 × 4, and their reversed orders). Thus, these problems are probably represented in memory as facts just as are the problems with double-digit products. We decided not to analyze these problems along with the problems with double-digit products, however, for two reasons. First, there was relatively little speedup for these problems with practice. This observation stands in contrast to problems with double-digit products, for which there was substantial speedup. We did not want problems such as 2 × 3 and 2 × 4 to mask the speedup present for problems with larger operands. Second, if we analyzed problems with double-digit and single-digit products separately, then the motor response requirements would be grossly the same for all problems within each arithmetic operation. For example, for the group of problems with double-digit products, all multiplication problems had double-digit answers and all division problems had single-digit answers.

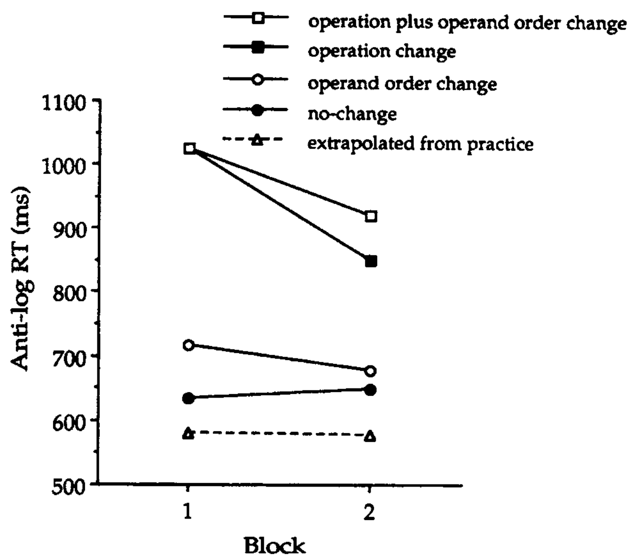

Figure 2. Antilog response time (RT) for correctly solved multiplication problems in Experiment 1 as a function of test condition and block.

Inspection of Figure 2 suggests that performance expected by extrapolating from practice was better than performance actually observed in the no-change condition. A 2 × 2 within-subjects ANOVA with condition (no change vs. extrapolated) and block (1 or 2) as variables was performed on the log RT to investigate the reliability of this effect. Actual performance in the no-change condition was indeed reliably worse than would be expected by extrapolating from practice, $F(1, 11) = 20.2$, $MS_e = .00114$. There were no effects involving block or the interaction of block and condition ($Fs < 1.0$).

A separate 2 × 2 × 2 within-subjects ANOVA with block (1 or 2), operation (same or different), and operand order (same or different) as variables was performed on the test log RTs. There were reliable effects of operation, $F(1, 11) = 35.7$, $MS_e = .01585$, operand order, $F(1, 11) = 5.5$, $MS_e = .00307$, and block, $F(1, 11) = 13.7$, $MS_e = .00217$. There was a reliable interaction of operation and block, $F(1, 11) = 14.0$, $MS_e = .00142$, indicating more speedup from Block 1 to Block 2 for different operation problems than for same operation problems. There was no reliable interaction between operation and operand order. A more detailed post hoc investigation of the operand order effect, however, showed it to be reliable for same operation problems, $F(1, 11) = 6.2$, $MS_e = .00251$, but not for different operation problems, $F(1, 11) = 1.3$, $MS_e = .00256$, indicating that the main effect of operand order may be primarily attributable to the difference between the no-change and operand order change conditions (this conclusion is consistent with the error results, which show virtually no difference between the operation and operation plus operand order change conditions).

It is important to consider possible effects that improvement in motoric (production) aspects of the task during practice might have had on overall performance in the various test conditions. Because the specific sequence of digits corresponding to the answers to problems in the same operation conditions (with or without order changes) were entered repeatedly on the keypad during practice, whereas, in most cases, the sequence corresponding to answers to problems in the different operation conditions were not, some of the RT advantage for same operation problems might be attributable to faster execution of the motor response for these problems. To investigate this possibility, we performed supplementary analyses on problems that allowed us to control for possible differences in motor RT among the various conditions. Specifically, three pairs of problems used across the practice sets shared the same product (2 × 6 and 3 × 4; 2 × 9 and 3 × 6; 3 × 8 and 4 × 6). The practice sets were constructed such that one member from each pair was presented as a multiplication problem in each practice set, and the other member was presented as a division problem. Thus, when the member of each pair that was presented as a division problem at practice was presented as a multiplication problem (in either operand order) at test, the subject was already experienced at executing the motor response for the answer by way of the other member of the pair that was presented as a multiplication problem at practice. An analysis restricted to these test problems thus equates problems in all conditions with respect to the subject's experience in entering the digit sequences that correspond to the answers. Consistent with the overall results, this analyses showed a reliable main effect of operation, $F(1, 11) = 12.5$, $MS_e = .02635$. The overall difference in antilog RT between the same and different operation conditions (collapsing across operand order and block) in this analysis was 289 ms, nearly identical to the comparable value in the overall analysis (282 ms), strongly indicating that specific experience with motor sequences made little if any contribution to the RT differences among the conditions at test.

Test: Division. As there were no reliable differences in error patterns in Blocks 1 and 2, the error data were collapsed across this variable. The error proportions for problems in the no-change, operand order change, operation change, and operation plus operand order change conditions were .022, .105, .109, and .093, respectively. A 2 × 2 within subjects ANOVA with operand order and operation as variables revealed a reliable effect of operation, $F(1, 11) = 5.2$, $MS_e = .01823$, and operand order, $F(1, 11) = 5.0$, $MS_e = .1485$. These effects were qualified by a reliable interaction of these variables, $F(1, 11) = 9.5$, $MS_e = .01349$; the error proportion in the no-change condition was especially low relative to other conditions. Post hoc pairwise comparisons showed no reliable differences among the operand order, operation, and operation plus operand order conditions (all $Fs < 1.0$).

The antilog of the mean log initiate RTs for correctly solved division problems are shown in Figure 3, including the predicted RT for problems in the no-change condition extrapolating from the power law fits to the division practice data. A 2 × 2 ANOVA, equivalent to that discussed for multiplication, revealed reliably slower performance in the no-change condition than would be expected by extrapolating from practice, $F(1, 11) = 20.3$ $MS_e = .00114$. There was no significant main effect of block in this analysis, nor was there a reliable interaction ($Fs < 1.0$).

In a 2 × 2 × 2 within-subjects ANOVA (with operation, operand order, and block as variables) comparing log RT among the test conditions, there were reliable main effects of operation, $F(1, 11) = 44.5$, $MS_e = .00466$, operand order, $F(1, 11) = 26.4$, $MS_e = .00495$, and block $F(1, 11) = 8.2$, $MS_e = .00176$. There was a two-way interaction between operation and block, $F(1, 11) = 12.8$, $MS_e = .00119$, indicating more speedup across blocks for different than for same operation conditions. There was also a two-way interaction between operation and operand order, $F(1, 11) = 19.5$, $MS_e = .00436$, reflecting the greater effect of a change in operand order in the same operation conditions than in the different operation conditions. Post hoc pairwise comparisons showed no differences among the operand order, operation, and operation plus operand order conditions (collapsing across block): $F(1, 11) = .4, 2.7, 1.0$, $MS_e = .00628, .00490, .00433$, for the operation versus operation plus operand order, operand order versus operation plus operand order, and operation versus operand order comparisons, respectively. These results reflect the pattern in Figure 3 in which the most salient effect is overall better performance in the no-change condition relative to all the other conditions.

A balanced check for possible motoric-related variations in RT across conditions was not possible for division problems as it was for multiplication problems. Note, however, that answers to division problems were all single digits, and each subject entered each digit many times across the entire set of multiplication and division problems during practice. This fact, combined with the fact that no motoric-related RT differences were present for multiplication, strongly indicates that any minor differences in the familiarity of the motoric responses played a negligible role in the RT patterns obtained across division conditions.

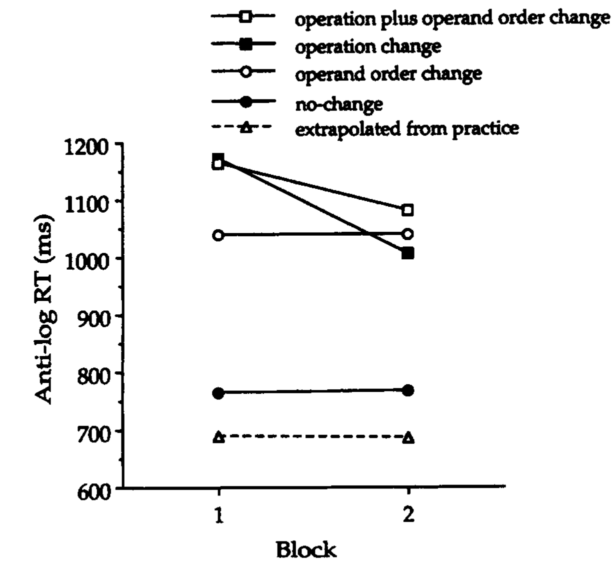

Figure 3. Antilog response time (RT) for correctly solved division problems in Experiment 1 as a function of test condition and block.

Discussion

Practice. There was substantial improvement in RT with practice for both multiplication and division, and the course of speedup was well described by a power law. A good power law fit does not, per se, rule out other possible mathematical descriptions of learning, such as the often considered exponential function. Rickard (1992) performed nonlinear regressions on these practice data and compared an extended version of the power law function (including parameters allowing for nonzero asymptotic RT and prior learning) with a similar version of the exponential function and found evidence favoring the power law. This finding, and those of other researchers that have demonstrated power function speedup (Crossman, 1959; Charness & Campbell, 1988; Staszewski, 1988), has direct relevance for models of mental arithmetic performance and skill acquisition. Most of the modeling efforts to date (e.g., Campbell & Oliphant, 1992; McCloskey & Lindemann, 1992; Rickard et al., 1992) focus on stationary performance and do not directly address the functional form of skill acquisition with practice. It is pervasively true in these models, however, that practice is assumed to strengthen representations or the connections among relevant representations. Ultimately, these and any future models will need to incorporate detailed learning algorithms explicitly stating both the mechanisms that govern the strengthening of representations and connections and the impact that this strengthening has on RT. Demonstrations of power law speedup thus provide important data against which to test candidate learning algorithms.

There was a reliable operation effect, an RT advantage for multiplication over division, both at the beginning and at the end of practice. This effect may exist because in formal education or in everyday life, multiplication is performed more frequently than is division, leading to greater skill or faster access to problem representations. This account is analogous to the often suggested frequency account of the problem size effect; the finding that problems with small operands, such as 3×4, are generally solved faster and more accurately than problems with large operands, such as 7×9 (see Ashcraft, 1992, for a review). At the moment, we are concerned primarily with an alternative possibility, namely, that the multiplication advantage might simply reflect the fact that the symbol $\times$ was used for both multiplication and division problems. Because the $\times$ symbol directly implies multiplication, its presence may have facilitated performance on multiplication problems, or interfered with performance on division problems, or both. This factor requires investigation before any other hypothesis can be seriously considered. Such an investigation was one purpose of Experiment 2, in which symbol ($\times$ and $\div$) was treated orthogonally to the actual operation (multiplication and division) required by a problem.

Test. Multiplication and division RTs in the no-change condition were reliably slower than expected by extrapolating from practice. Note that this finding could not reflect forgetting because the test was given immediately after the last block of practice. One account of this finding, consistent with Battig's (1979) contextual interference hypothesis, is that exposure to the various altered problems at test resulted in the activation of problem representations that were not active during practice. These newly active representations may then have competed in the retrieval process, slowing retrieval times for practiced problems (Campbell, 1987a). An alternative account suggested by Campbell is that subjects may strategically allow more processing time for problems at test to bring overall error rates more in line with the low error rates during practice. It is not possible to differentiate strongly between these two accounts on the basis of data from Experiment 1.

The relative performance among the test conditions can be summarized straightforwardly: When the presented numbers (disregarding operand order) and the formally required arithmetic operation of a test problem were exactly the same as those of a problem solved during practice (as in the no-change conditions for both operations and the operand order change condition for multiplication), performance was relatively good. When the presented numbers and the required operation of a test problem did not completely match those of a practice problem (as in the operation and operation plus operand order change conditions for both operations and the operand order change condition for division), performance suffered substantially; error rates were three to five times higher, and RTs were more than 300 ms slower.

The foregoing summary suggests a working model of skilled arithmetic knowledge that we refer to as an *identical elements* model, according to which there is a single and functionally distinct unit of arithmetic knowledge corresponding to each unique combination of the two numbers (ignoring order) that constitute a problem (e.g., 4 and 7), the number that is the answer (e.g., 28), and the arithmetic operation formally required to produce the answer (e.g., multiply).[3] Note that "the operation formally required" refers to the operation required in the mathematical sense, rather than to the arithmetic symbol present in the problem. For example, the answer to $28 = \_\_ \times 4$ requires division. The model assumes distinct perceptual, cognitive, and motor stages of arithmetic fact retrieval (see also McCloskey et al., 1985) and it applies to the structure of knowledge as represented within the cognitive stage. Numbers are treated as abstract elements superordinate to the perceptual characteristics of the modality or physical format in which problems are presented.[4] Problems that have exactly the same elements will access the same knowledge unit within the cognitive stage, despite any perceptual differences, such as format or modality of presentation. For example, a problem presented in numerical format, such as 4×7, and the same problem presented in a written verbal format, such as "four times seven," will access the same knowledge unit. Similarly, multiplication problems that differ only in operand order, such as 3×8 and 8×3, will access the same knowledge unit. Indeed, any problems that differ only with respect to detailed characteristics of the format (e.g., horizontal vs. vertical presentation, variations in the symbol used to denote an operation) will access the same knowledge unit. In contrast, problems that differ with respect to even one element will access completely different knowledge units. So, for example, complementary problems from two operations (e.g., $4 \times 7 = \_\_$ and $4 \times \_\_ = 28$), and related problems within a noncommutative operation (e.g., $28 = \_\_ \times 4$ and $28 = \_\_ \times 7$), access completely different knowledge units.

Because practice will strengthen only the knowledge units corresponding to the practiced problem, the identical elements model makes straightforward predictions about positive *transfer of learning* to the various test conditions. If the elements of the test problem match exactly with the elements of a problem seen during practice, there will be substantial positive transfer of learning to the test problem, despite any other problem differences. Note, though, that the model does not necessarily predict total transfer of learning in this case because some perceptual processing advantage could accrue for the practiced problem. Thus, for example, the model is not necessarily inconsistent with the slight but reliable increase in RT that was observed in the operand order change condition relative to the no-change condition for multiplication, here and in the previous study by Fendrich et al. (1993): It could be the case that a perceptual advantage accrued for the practiced operand order, but that the same knowledge units were accessed for corresponding problems in the two conditions. In contrast, if the elements of the test problem do not exactly match the elements of at least one practice problem, and if there are no general transfer effects from practice to test (e.g., no improvements with practice in general perceptual or motor processes), then the identical elements model makes a prediction of absolutely no positive transfer. Even if there is some general transfer, the identical elements model predicts substantially poorer performance when the elements of the test problem do not exactly match those of a practice problem. Consistent with the prediction, performance in the operation change and operation plus operand order change conditions for both operations, and in the operand order change condition for division, was much poorer than performance in the other conditions.

If the knowledge units specified by the identical elements model are interpreted as being completely independent of one another, then the model also predicts no negative transfer of learning under any circumstances. The negative transfer results of Campbell (1987a), however, show this prediction to be incorrect. An alternative interpretation of the model, which we adopt, is that access to the knowledge unit corresponding

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[3] The reader may question whether one or more of these elements is redundant. The model, however, would make substantially different and often untenable predictions if any one of the elements that we have defined were excluded. For example, if the answer were excluded as an element, the model would predict that the same knowledge unit would be accessed by $28 \div 4 = \_\_$ and $4 \div 28 = \_\_$. Note also that the model can be restated equivalently in terms of the two numbers presented in the problem, the formal operation to be performed, and, if the operation is noncommutative, the order of the presented numbers.

[4] Although our use of the term *identical elements* is similar to and strongly motivated by its previous uses by Thorndike (1906) and Singley and Anderson (1989), there are also some differences. Thorndike's elements were the stimulus items themselves. Taken literally, this position predicts no transfer any time there is any change whatsoever in the makeup of the stimulus items (such as a change in operand order). Our elements are abstract representations of numbers and arithmetic operations. Singley and Anderson also proposed a general abstract identical elements model of skill acquisition and transfer, framed within the ACT* architecture (Anderson, 1983). However, arithmetic facts could potentially be modeled using the ACT* architecture in many different ways that would yield different predictions about transfer. In contrast, our identical elements model, although much narrower in scope, makes specific predictions regarding mental arithmetic.

exactly to a given problem is the only direct retrieval route to correct performance on that problem. Speedup and reduction in error rate with practice on a given problem reflect more efficient processing within the corresponding knowledge unit. We leave open the possibility, however, that knowledge units not corresponding to a given problem can potentially produce contextual interference (Battig, 1979) through some unidentified mechanism external to the model. Thus, for example, practice on one set of problems can negatively influence performance on another set of problems corresponding to different knowledge units, as in the Campbell (1987a) study. In summary, the identical elements model predicts substantial, although not necessarily total, positive transfer whenever the elements of the test problem match exactly with those of a practice problem. The model predicts no transfer or negative transfer whenever (a) the elements of the test problem do not match exactly with those of a practice problem and (b) general positive transfer effects (e.g., general perceptual or motor speedup with practice) are either negligible or can be factored out.

Table 3
Examples of Each of the Four Problem Types Seen at Practice and the Corresponding Test Conditions in Experiment 2

	Test condition			
Practice	No change	Symbol change	Operation change	Operation and symbol change
____ = 4 × 7	____ = 4 × 7	____ ÷ 4 = 7	28 = __ × 7	28 ÷ __ = 7
____ ÷ 9 = 5	____ ÷ 9 = 5	____ = 9 × 5	45 ÷ __ = 5	45 = __ × 6
48 = __ × 6	48 = __ × 6	48 ÷ __ = 6	____ = 8 × 6	____ ÷ 8 = 6
18 ÷ __ = 3	18 ÷ __ = 3	18 = __ × 3	____ ÷ 6 = 3	____ × 6 = 3

Note. From top to bottom, the four types of problems at practice represented above are multiplication with symbol "×", multiplication with symbol "÷", division with symbol " ×", and division with symbol "÷".

Experiment 2

In this experiment, we examine practice–transfer effects manipulating operation (multiplication and division, as in Experiment 1) and a new variable, symbol (see Table 3), in a design that allows us to explore two issues raised by Experiment 1. First, in Experiment 1 there was an operation effect in the practice data, a performance advantage for multiplication over division. As we discussed earlier, it is possible that use of the multiplication symbol × for all problems in Experiment 1 is responsible for this effect. Performance may be sensitive to the degree of consistency between the mathematical operation required and the conventional symbol used. The orthogonal manipulation of operation and symbol in this experiment allows us to examine the possibility of a consistency effect.

Second, the test conditions allow for additional tests of the identical elements model. Performance was evaluated on each of the practice problems, as well as on several altered versions of each practice problem, reflecting a change in symbol, a change in operation, or a change in both symbol and operation (see Table 3). On the basis of the model, we expect a substantial overall performance advantage for all problems in the same operation conditions (i.e., the no-change and symbol change conditions), for which the elements match exactly with those of a problem seen during practice, relative to problems in the different operation conditions (i.e., the operation and operation plus symbol change conditions), for which the elements do not match exactly with those of a problem seen at practice.

Note that it is not clear a priori whether, as predicted by the model, subjects will treat problems that differ only in terms of the arithmetic symbol as the same problem. For example, subjects may interpret ____ = 4 × 7 as "what is four times seven?" whereas they may interpret ____ ÷ 4 = 7 as "what divided by four is seven?" Similarly, subjects may interpret 28 = __ × 7 as "twenty-eight equals what times seven?" whereas they may interpret 28 ÷ __ = 7" as "twenty-eight divided by what equals seven?" It is quite possible that the knowledge structures that are accessed and presumably strengthened with practice are strongly dependent on these potential differences in how the problems are interpreted. The identical elements model, however, predicts that the same knowledge structure is accessed regardless of symbol, and thus it predicts substantial transfer of learning to symbol change problems at test.

Method

Subjects. Twelve subjects from an introductory psychology course received credit for participating in the experiment.

Materials and procedure. The materials and procedures were the same as those in Experiment 1, with the following exceptions: Half of the multiplication and division problems were expressed with the symbol × and half with the symbol ÷. Thus, a practice set in this experiment can be derived from Table 2 by switching the symbol to ÷ for half of the multiplication and half of the division problems. This manipulation yielded four problem types at practice; multiplication problems with symbol ×, multiplication problems with symbol ÷, division problems with symbol ×, and division problems with symbol ÷. On the immediate and delayed tests, each problem was presented again exactly as it was at practice (the no-change condition), with a change in symbol, with a change in operation, and with a change in both operation and symbol. Only one operand order was presented for each problem across both the practice and test phases of the experiment.

Results and Discussion

Practice. Error proportions for the four problem types were as follows: For multiplication problems with symbol ×, .053, for multiplication problems with symbol ÷, .053, for division problems with symbol ×, .030, and for division problems with symbol ÷, .052.

In Figure 4, the log initiate RT (averaged across subjects and correctly solved problems) is plotted as a function of log block and problem format, with regression fits shown separately for each of the four problem formats. We analyzed these data using log block as a continuous within-subjects variable, and operation and symbol as categorical within-subjects variable. The overall r^2 was .85. There was a large main effect of log block, $F(1, 11) = 161.1$, $MS_e = .01963$, reflecting an overall improvement in log RT with practice, and a reliable interac-

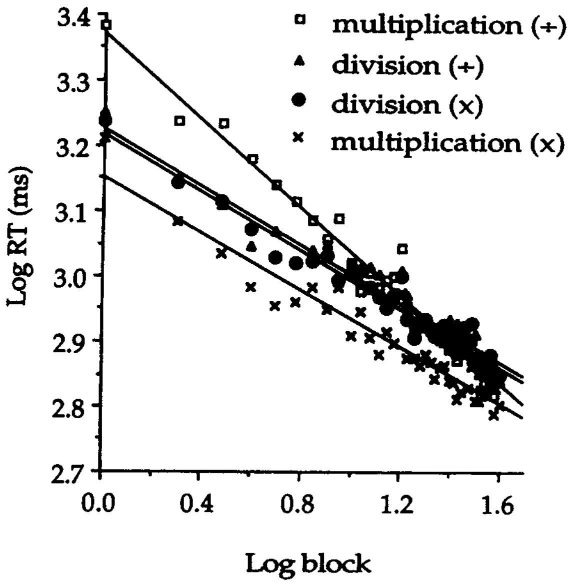

Figure 4. Log response time (RT) for all correctly solved practice problems in Experiment 2 plotted as a function of log block and problem format. Multiplication (×) = multiplication problems with symbol ×; multiplication (÷) = multiplication problems with symbol ÷; division (×) = division problems with symbol ×; division (÷) = division problems with symbol ÷.

tion of log block and symbol, $F(1, 11) = 28.8$, $MS_e = .01623$. Both of these effects were qualified by a three-way interaction among log block, symbol, and operation, $F(1, 11) = 19.7$, $MS_e = .01193$. As shown in Figure 4, three of the four problem types exhibit essentially the same slope over blocks, whereas the slope of the fourth problem type (multiplication with symbol ÷) was much steeper. There is no obvious explanation for this difference other than the relative unfamiliarity of the multiplication with symbol ÷ format (e.g., ____ ÷ 6 = 7) for presenting arithmetic problems. We suspect that early during practice, there was some delay for multiplication problems with symbol ÷ because of a need to interpret, or decode, the problem deliberately to determine, in terms of the identical elements model, the knowledge unit to be accessed. This interpretive stage may have been less prominent or completely absent for the other formats because of their relative familiarity.

Two effects not involving block were significant at the beginning (first block) of practice. There was a main effect of symbol, $F(1, 11) = 31.5$, $MS_e = .0326$, and also an interaction between operation and symbol, $F(1, 11) = 12.3$, $MS_e = .03259$. Both of these effects reflect the exceptionally poor performance at the beginning of practice on multiplication problems with symbol ÷. There was no reliable effect of operation, $F(1, 11) < 1.0$. The results from the end (last block) of practice looked different. There was a reliable effect of operation, $F(1, 11) = 8.8$, $MS_e = .03809$, reflecting faster performance on multiplication problems, although the effect of symbol, $F(1, 11) = 1.9$, $MS_e = .05445$, and the interaction of operation and symbol, $F(1, 11) < 1.0$, were no longer significant.

As we noted earlier, one purpose of this experiment was to determine whether the operation effect observed in Experiment 1 reflected a fundamental performance advantage for multiplication, or, rather, simply the fact that the symbol for multiplication, ×, was used for both multiplication and division problems. The findings from Experiment 2 support the hypothesis that the operation effect reflects a fundamental advantage for multiplication. Despite the poor performance for multiplication problems with symbol "÷" at the beginning of practice, by the end of practice there was a reliable overall RT advantage for multiplication problems over division problems, and average RTs were roughly equivalent for problems that differed only in the symbol.

Test. Problem format (each of the four combinations of operation and symbol are defined here as a separate format) did not enter into any reliable interactions with the other test variables, and thus we collapsed across this variable for all reported test analyses. The error proportions, collapsed across block, were .042, .042, .100, and .090 for the no-change, symbol change, operation change, and operation plus symbol change condition, respectively. As with Experiment 1, there was a large increase in the error proportion with a change in operation. A 2 × 2 within-subjects ANOVA with symbol and operation as variables showed a reliable effect of operation, $F(1, 11) = 21.8$, $MS_e = .00726$, but neither symbol, $F(1, 11) = 1.2$, $MS_e = .00436$, nor the interaction of operation and symbol, $F(1, 11) < 1.0$, was statistically significant.

The antilog of the mean log initiate RT (averaged across problems and subjects) is plotted in Figure 5 by block (1 or 2) and test condition (no-change, symbol change, operation change, and symbol plus operation change). The expected RTs in the no-change condition extrapolating from a power function fit to the overall practice data are also shown. An ANOVA comparing no-change to extrapolated values (see the *Results* section in Experiment 1 for details) revealed reliably slower RT in the no-change condition than would be expected by extrapolating from practice, $F(1, 11) = 8.2$, $MS_e = .00067$. There were also reliable effects of block, $F(1, 11) = 7.9$, $MS_e =$

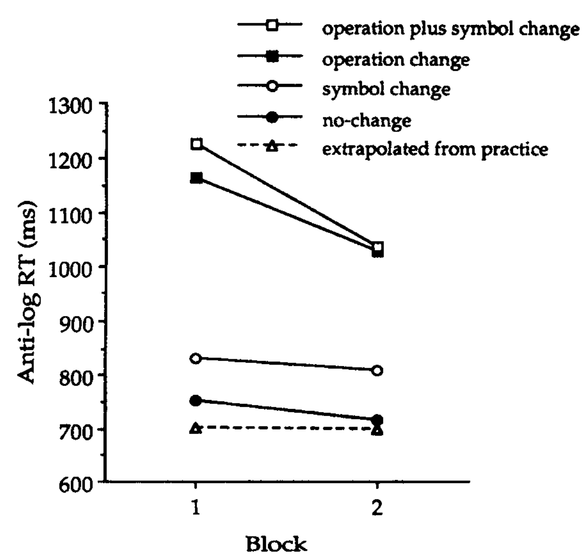

Figure 5. Antilog response time (RT) for correctly solved problems in Experiment 2 as a function of test condition and block.

.00023, and the interaction of block and condition, $F(1, 11) = 5.4$, $MS_e = .00021$, reflecting more speedup from Block 1 to 2 for the no-change condition than would be expected by extrapolating from practice. Inspection of Figure 5, however, shows this interaction to be of relatively small magnitude. The primary result here is the slower performance in the no-change condition than would be expected by extrapolating from practice, which replicates the results of Experiment 1.

A 2 × 2 × 2 ANOVA with within-subjects variables of block (1 or 2), operation (same or different), and symbol (same or different) was performed on the log initiate RT for correctly solved test problems. Consistent with the predictions of the identical elements model, there was a strong effect of operation, $F(11, 1) = 132.9$, $MS_e = .00433$. There was a main effect of block, $F(1, 11) = 56.3$, $MS_e = .00069$, and also an interaction of operation and block, $F(1, 11) = 45.1$, $MS_e = .00028$, indicating more speedup from Blocks 1 to 2 for different-operation than for same-operation problems. There was also a main effect of symbol, $F(11, 1) = 26.8$, $MS_e = .00083$, reflecting better performance in the same symbol conditions than in the different symbol conditions. There was no reliable interaction between operation and symbol. Further examination of the symbol effect, however, showed a reliable difference between the no-change and symbol change conditions $F(1, 11) = 13.4$, $MS_e = .00204$, but no reliable differences between the operation change and operation plus symbol change conditions, $F(1, 11) = 4.6$, $MS_e = .00044$.

In summary, as predicted by the identical elements model, the primary determinant of performance was whether or not the elements of a test problem were the same as the elements of a practice problems; that is, performance was relatively good in the same operation (no-change and symbol change) conditions and relatively poor in the different operation (operation change and operation plus symbol change) conditions.

General Discussion

The most straightforward account of the operation effect observed in the practice data of Experiments 1 and 2 is that subjects came into the experiments having been exposed more frequently to multiplication problems than to division problems. There are two theoretically distinct ways in which a frequency advantage could give rise to better multiplication performance. First, more frequent exposure might simply speed retrieval of multiplication facts. Because multiplication and division problems were presented with equal frequency during practice in these experiments, the advantage for multiplication might decrease (but not completely disappear) by the end of practice. Alternatively, division facts might be too weakly represented to support direct retrieval, leading subjects to use a backup strategy (see Siegler, 1988) for solving these problems. One such strategy is mediation of division by way of better developed multiplication knowledge. For example, for the problem 42 = ____ × 6, subjects might adopt a strategy of plugging in candidate answers, performing the multiplication, and then checking the result with the provided answer. If such strategies do account for the operation effect at the beginning of the experiments, then there are two obvious possibilities for the effects of practice on division. First, practice might strengthen the division facts sufficiently for direct retrieval to take over at some point. Second, the mediational processes might simply become more efficient or automatized with practice.

We cannot differentiate strongly between these possibilities with data from the current experiments, but there at least two lines of evidence suggesting direct retrieval for division, at least toward the end of practice. First, as discussed earlier, there is evidence in the arithmetic literature that skilled multiplication typically involves direct retrieval from a network of facts, even when algorithms can be demonstrated during initial learning (Siegler, 1988). It is therefore reasonable to assume that division performance reflects fact retrieval as well, given sufficient practice. Second, mean RTs (as predicted by the power function fits) for multiplication and division problems on the last of block of practice in Experiment 1 were 580 ms and 689 ms, respectively. These RTs are too fast to allow much in the way of mediational processes, and the difference between the multiplication and division RTs is certainly too small to support an hypothesis of direct retrieval for multiplication and frequent mediation for division.

All of the major differences among the test conditions of both experiments are predicted by the identical elements model. The model predicts substantial positive transfer of learning when the elements of a test problem match exactly the elements of a problem seen during practice. This prediction was confirmed by the clear evidence of positive transfer to problems in the no-change conditions across both experiments, in the operand order change condition for multiplication of Experiment 1, and in the symbol change condition of Experiment 2. In contrast, the model predicts no positive transfer of learning when the elements of the test problem do not match exactly with those of a practice problem. Consistent with this prediction, performance levels were substantially lower in Experiment 1 for the operation and operation plus operand order change conditions with both multiplication and division, and the operand order change condition with division, and in Experiment 2 for the operation and operation plus symbol change conditions.

Two other, less prominent differences among the test conditions are at least potentially consistent with, even if not directly predicted by, the model. First, there were reliable performance advantages in terms of RTs (but not error proportions) in the no-change conditions of both experiments relative to the operand order change condition (Experiment 1) and the symbol change condition (Experiment 2). Both of these findings are consistent with the model, assuming that they reflect greater fluency of perceptual processing for the no-change conditions. An experimental approach to testing this assumption is discussed later. Second, there were trends in the test data suggesting slightly better performance in the operation change condition relative to the operation plus operand order change condition for both multiplication and division in Experiment 1, and also slightly better performance in the operation change condition relative to the operation plus symbol change condition in Experiment 2. These relatively small effects can potentially also be accounted for by assuming a perceptual advantage for problems in the operation change conditions, analogous to the perceptual advantage suggested previously for no-change problems. Recall that the answer appeared on the screen as the subject typed it, replacing the answer blank. For example, after a subject typed the answer to ___ = 4 × 7, the problem-answer combination briefly appeared as 28 = 4 × 7. This perceptual exposure to the entire problem-answer configuration could have resulted in slightly faster performance at test on operation change problems, like 28 = __ × 7, which are partial reinstatements of the same configuration, than on operation plus operand order change problems, such as 28 = __ × 4, or operation plus symbol change problems, such as 28 ÷ __ = 7. The fact that there were no reliable effects of operand order or symbol in the analysis of error proportions is consistent with the perceptual account of the effects of these variables in the RT analyses: RTs presumably reflect both perceptual and cognitive (i.e., fact retrieval) related processing, but errors may be largely, if not completely, driven by fact retrieval processes.

A stronger test of the identical elements' prediction of no positive transfer when the elements of the test problem fail to match those of a practice problem would require pretesting subjects on a group of problems, practicing subjects on a subset of these problems, and then administering a posttest on all problems. If general speedup factors are negligible or can be factored out, and if none of the problems on which subjects practice have elements exactly identical to those on which they did not practice, then the identical elements model predicts no positive transfer to unpracticed problems at posttest. That is, performance on unpracticed problems should be no better at posttest than at pretest. The results of the Campbell (1987a) study, reviewed earlier, are consistent with this prediction.

The issue addressed in Campbell's (1987a) study was whether there would be positive transfer to unpracticed problems within a given operation (multiplication). An analogous question in Experiments 1 and 2 is whether there is any positive transfer to complementary problems in a new operation. Although we did not have a pretest in either experiment, the problems seen during practice and test were completely counterbalanced. Thus, it is reasonable to compare performance at the beginning of practice with performance at test on what we will term *new* problems (i.e., problems in the operation plus operand order change condition in Experiment 1 and in the operation plus symbol change condition in Experiment 2). The identical elements model predicts no positive transfer to these problems if there is no general transfer (i.e., transfer not related to specific problems on which subjects practiced) from practice to test. In Experiment 1, the overall error proportions and antilog mean RTs were .051 and 1,278 ms on the first block of practice and .103 and 1,094 ms for new problems on the first block of test. In Experiment 2, the corresponding values were .080 and 1,883 ms on the first block of practice and .096 and 1,208 ms on the first block of test.

These results provide mixed support for the model. Negative transfer on the error proportions (−.052 and −.016 in Experiments 1 and 2, respectively) is consistent with the model, but the finding of positive transfer on the RTs (184 ms and 675 ms in Experiments 1 and 2, respectively) appears, on the surface at least, to be inconsistent with the model. There are, however, at least two ways to interpret results that are consistent with model. First, the negative transfer indicated by the error proportions and the positive transfer indicated by the RTs might reflect a shift in the subjects' speed-accuracy criterion from the beginning of practice to test. Subjects might have been more accuracy oriented at the beginning of practice and more speed oriented at test. This interpretation leaves open the possibility of no net transfer of learning to new problems at test.

An alternative interpretation is that the negative transfer on error proportions reflects interference because of learning that took place during practice, but that the positive transfer on RTs reflects more general learning that is not directly related to fact retrieval within the cognitive stage. Consistent with the interpretation that the negative error transfer reflects interference, Rickard (1992) reported detailed error analyses on the Experiment 1 test data and found that more than 70% of errors for multiplication problems in the operation and operation plus operand order change conditions at test were correct answers to table-related problems seen during practice. Analogous results were also obtained for division. These percentages were reliably larger than would be expected by chance.

There are many potential sources of positive RT transfer in these experiments that would involve general processes not directly tied to fact retrieval, including general speedup in perceptual processing and motor response.[5] In addition, there is evidence of substantially improved efficiency at test, relative to the beginning of practice, in processing the various formats in which the problems were presented. First, note that the amount of positive transfer from the beginning of practice to new problems at test in Experiment 2 (675 ms) was much larger than that in Experiment 1 (184 ms). This effect primarily reflects a large performance disadvantage (605 ms) for Experiment 2 relative to Experiment 1 at the beginning of practice. This disadvantage was reduced to 114 ms at test. These results suggest a "format level" contextual interference (Battig, 1979) account of much of the positive RT transfer in Experiment 2. Assuming that the number of different formats encountered constitutes a form of context in these experiments, then the concurrent presence of four unique problem formats in Experiment 2 might have slowed processing of all problems relative to the less contextually varied case of Experiment 1. By test, however, all subjects were familiar with all formats, and any performance disadvantage attributable to the more varied formats in Experiment 2 would likely be substantially reduced. Note additionally that a similar increase in the efficiency of format level processing during practice can potentially account for the positive RT transfer observed in Experiment 1.

[5] The motor response speedup was estimated by fitting a power function to the latency between the pressing of the first digit of the answer and the pressing of the second digit of the answer for multiplication problems in Experiment 1. A speedup of 47 ms from the first to the last block of practice was indicated by this analysis. The r^2 was .89. This measure should provide a relatively pure measure of motor response speedup (i.e., assuming that the entire answer is retrieved before executing the motor response, this measure is unlikely to reflect either encoding or fact retrieval related processes). Note also that the estimate of 47 ms is, if anything, an underestimate; much of the motor response for the second digit of the answer could be preprogramed before entering the first digit (see Card, Moran, & Newell, 1983).

Second, the mean RT for multiplication problems with symbol ÷ in Experiment 2 on the first block of practice was more than 800 ms slower than mean RT for any other problem format in either experiment. On new problems at test, however, the mean RT for this condition was 1,293 ms, only 114 ms slower than the overall mean RTs of the other three conditions in Experiment 2. Clearly, then, much of the speedup that occurred for these problems during practice reflects a general improved ability to interpret the cognitive processing required by these problems (e.g., the operation to be performed).

In summary, then, the identical elements model provides at least a good first order approximation to the organization of knowledge for the basic arithmetic facts. Because the identical elements model makes straightforward and testable predictions, we believe that it provides a useful starting point for future investigations into the detailed structure of arithmetic knowledge. To make this point more concrete, we outline briefly two predictions of the model that will be tested in future experiments. As discussed above, the model predicts no positive transfer with a change in operation from practice to test, given that general transfer effects can be factored out. A strong test of this prediction would involve replicating the essential features of Experiment 1, but with an additional new problems condition at test for both multiplication and division. These new problems would not be seen during practice in either operation. Because the various general transfer factors discussed above would be equated for new and operation plus operand order change conditions at test, the identical elements model predicts that performance in these conditions will not differ. Another prediction of the identical elements model is that perceptual familiarity alone is responsible for the advantage of no-change problems over (a) operand order change problems for multiplication in Experiment 1 and (b) symbol change problems in Experiment 2. One test of this prediction for operand order transfer would involve practicing subjects on several problems each in only one operand order, with some problems presented in numerical format (e.g., 4 × 7), and some problems presented in written verbal format (e.g., five times eight), and then testing them on each practice problem presented in both formats and in both operand orders, and on new problems not seen during practice. The identical elements model makes two predictions. First, within a given problem format at test, there should be a substantial performance advantage for all old (practice) problems over new problems, even when the format, operand order, or both is changed from practice to test. Second, any performance advantage for no-change over operand order change problems should disappear when these problems are presented in a novel format. This prediction follows from the model because processing of same and reversed operand order problems presented in the novel format should not benefit differentially from perceptual familiarity acquired during practice.

Implications for the Network-Interference and MATHNET Models

Our empirical results bear on the viability of several assumptions embodied in both the network-interference (Campbell & Oliphant, 1992) and the MATHNET (McCloskey & Lindemann, 1992) models of mental arithmetic. Before beginning this discussion, we note that the symbol manipulation of Experiment 2 addresses issues that appear to be outside the scope of phenomena toward which either of these models have been directed to date (although it is certainly possible that the models could be extended to cover these issues). Thus, we focus on issues related to the operand order and operation manipulations of Experiment 1.

Campbell and Oliphant (1992) conjectured that their model could easily be extended to division by assuming a single representation for corresponding multiplication and division problems. This approach, however, would not fare well in accounting for our results. As discussed earlier, such a model would predict substantial positive transfer of learning to the operation change conditions for both multiplication and division and better performance in the operation change condition than in the operand order change condition for multiplication. Clearly, though, performance in the operand order change condition for multiplication was much better than performance in the operation change condition. An alternative approach to extending the network-interference model to division would appear to be necessary to account for these results.

It is also unclear at present whether the network-interference model will be able to predict the strong (although not total) positive transfer of learning to operand order change problems for multiplication. In principle, the model should be able to account for this finding because representations for both orders of a problem can become active whenever either order is encountered, and thus the representation for the practiced order can potentially facilitate processing of a problem presented in the unpracticed order. Nevertheless, given the complexity of the model, simulations will be needed to demonstrate conclusively that it can indeed generate the operand order transfer effect.

As discussed in the introduction, the MATHNET model of McCloskey and Lindemann (1992) does not appear to predict the substantial transfer to operand order change problems for multiplication that was observed by Fendrich et al. (1993) and that was replicated in Experiment 1. McCloskey and Lindemann encountered a similar conflict between the predictions of their model and data that has been collected from patients with brain injuries. Whereas error rates for complementary operand orders were essentially uncorrelated in their simulations, these correlations among patients with brain injuries were typically positive and quite strong (McCloskey et al., 1991; Sokol et al., 1991). As one potential account for this discrepancy, McCloskey and Lindemann (1992) suggested that their patients may have mentally transformed (reversed) the operand order if they were unable to retrieve the answer with the given operand order. Such a strategy would yield the observed correlations because performance on both orders would be driven by the more intact, or accessible, representation. It is conceivable that a similar transformation process might account for the increase in RT with a change in operand order for multiplication in Experiment 1 and in the Fendrich et al. (1993) studies. The practiced operand order may become strong enough that, at test, the most efficient strategy for solving the reversed order problems is to transform the problem into the practiced order before retrieving the answer.

This account seems unlikely, however, because the RT advantage for the no-change condition over the operand order change condition for multiplication was only about 65 ms. The required transformation process, which would necessitate conscious mediation, is unlikely to take place in so little time. Thus, the MATHNET model, as currently formulated, does not appear to provide a viable account of the representation of operand order information in adults. It is more likely that either separate representations can both participate in retrieving the answer when either order is presented, as in the network-interference model, or that a single underlying representation mediates performance on both orders, as in the identical elements model.

Implications for the Development of Arithmetic Skills

It is of interest to contrast our findings with studies exploring the initial acquisition of arithmetic skills. For example, Siegler (1986) found evidence suggesting that children represent complementary operand orders independently. Similarly, Reder and Ritter (1992) trained college students on more complex multiplication problems (e.g., "17 × 24") and found evidence suggesting learning for their subjects was order specific. Thus, the structure of factual arithmetic knowledge appears to change, at least with respect to order information, as skill improves. Through some yet-to-be-identified generalization process, the cognitive distinction between complementary operand orders largely disappears by adulthood. Along similar lines, the current studies show little if any transfer of learning across operation for college subjects. It is a commonly held belief, however, that children rely on multiplication knowledge to learn division. If this belief is true, then in an important sense there is positive transfer of learning across operation, at least from multiplication to division, for less skilled subjects. In discussing potential accounts of the operation effect during practice in these experiments, we suggested that, during early practice, performance on some division problems may have been mediated by multiplication knowledge, with a transition to direct retrieval of division facts as skill improves. The practice interval in these studies may represent the tail end of the skill acquisition interval within which such mediation takes place. If these speculations are correct, then strong positive transfer from multiplication to division (but not necessarily the reverse) should be obtained for young children in a similar study, with decreasing degrees of transfer for older children for whom the initial level of division skill would be greater.

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1995 APA Convention *Call for Programs*

The *Call for Programs* for the 1995 APA annual convention appears in the September issue of the *APA Monitor.* The 1995 convention will be held in New York, New York, from August 11 through August 15. The deadline for submission of program and presentation proposals is December 2, 1994. Additional copies of the *Call* are available from the APA Convention Office, effective in September. As a reminder, agreement to participate in the APA convention is now presumed to convey permission for the presentation to be audiotaped if selected for taping. Any speaker or participant who does not wish his or her presentation to be audiotaped must notify the person submitting the program either at the time the invitation is extended or before the December 2 deadline for proposal submission.